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Nonlinear waves and solitons in physical systems

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Abstract

Advances in nonlinear science have been plentiful in recent years. In particular, interest in nonlinear wave propagation continues to grow, stimulated by new applications, such as fiber-optic communication systems, as well as the many classical unresolved issues of fluid dynamics. What is arguably the turning point for the modern perspective of nonlinear systems took place at Los Alamos over 40 years ago with the pioneering numerical simulations of Fermi, Pasta, and Ulam. A decade later, this research initiated the next major advance of Zabusky and Kruskal that motivated the revolution in completely integrable systems. With this in mind, the conference on Nonlinear Waves in Solitons in Physical Systems (NWSPS) was organized by the Center for Nonlinear Studies (CNLS) at Los Alamos National Laboratory in May of 1997, to assess the current state-of-the-art in this very active field. Papers from the conference attendees as well as from researchers unable to attend the conference were collected in this special volume of *Physica D*. In this paper, the contributions to the conference and to this special issue are reviewed, with an emphasis on the many unifying principles that all these works share. Copyright © 1998 Elsevier Science B.V.

1. Introduction

Most natural systems are nonlinear, and are therefore modeled by nonlinear systems of equations. The essential difference between linear and nonlinear systems is that linear systems satisfy a simple superposition principle. That is, any two solutions of a linear system, added together, form a new solution to the same equations; this is not the case for solutions of nonlinear systems. This superposition principle allows the solution of a linear problem to be broken into pieces, which are then solved independently by, for example, the Fourier or Laplace transform, and then added back to form a solution to the original problem.

Despite the difficulty caused by the lack of superposition principles, the last 40 years have seen revolutionary progress in solving nonlinear systems, guided by advances in experiments, phenomenal success in the computer simulation of nonlinear systems, and new mathematical analytical tools, such as the inverse spectral transform and methods based on Hamiltonian systems. The synergy between theory, computation, and experimental sciences continues to lead researchers to new levels of understanding.

One field of nonlinear science that has experienced some of the most spectacular progress is wave propagation phenomena. In this class of problems, asymptotic procedures that take advantage of small parameters in physical regimes of interest often result in a few "universal" partial differential equations (PDEs). It is one of the mysteries of nature that many of these equations turn out to be integrable, which essentially means that their solutions can be represented as a superposition of special wave modes. Thus some manifestly nonlinear problems can after all be

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broken down and solved via a nonlinear decomposition in a manner analogous to the way that linear wave equations are solved with Fourier or Laplace transforms. The most famous of these special wave modes are perhaps the solitons, localized waves that collide elastically, suffering only a shift in phase. This robustness of solitons to overcome strong perturbations is largely due to a subtle balance between the tendency of nonlinearity to increase the wave slope and the linear dispersion that tends to flatten a wave. The occurrence of this type of balance is widespread in fluid mechanics. The interplay between nonlinearity and dispersion selects distinctive patterns in natural systems in situations ranging from the great red spot on Jupiter to vortices in liquid helium.

Nature abounds with examples of nonlinear waves. The universality of the underlying mechanisms for nonlinear waves has allowed advances in nonlinear wave motion in fluid dynamics to be applied to molecular dynamics and nonlinear optics and vice versa. Nonlinear mechanisms that just a few years ago were considered unsolvable are now understood and are applied to create ultrahigh-speed optical transmission lines, to model ocean waves, and to better understand the behavior of conducting polymers, cavitons in plasma physics, and discrete lattices like Josephson-junction transmission lines.

Often, research on a given natural system began by investigating a one-dimensional PDE derived as an approximation to an experimental system in fluid dynamics, nonlinear optics, material science, or chemically reacting fluid. Many of these have known nonlinear wave solutions (which are sometimes soliton solutions). This limited knowledge is then extended with numerical simulations. Next, the models are enriched by adding new terms for physical effects, such as dissipation, driving terms, or higher-dimensional effects, which were initially left out of the model. Here, computation begins to play an even bigger role and often leads the analytic and experimental investigations.

This special issue of *Physica D* originated with the conference on Nonlinear Waves in Solitons in Physical Systems (NWSPS) organized by the Center for Nonlinear Studies (CNLS) at Los Alamos National Laboratory in May of 1997. Researchers unable to attend the conference were also encouraged to contribute to the special issue. In this work, we survey some of the papers presented at the meeting and those appearing in this special issue. We also cite some of talks and posters given at NWSPS in this introductory paper as it occurred to us to do so. We did not make an attempt to discuss all of the talks and posters, as this would go beyond the space we have available. We stress that omission of mention here is in no way meant to reflect negatively on the work but is rather a reflection of the interest bias that our own research areas inevitably introduce.

2. Fluid dynamics

Perhaps our first perception of wave motion comes from observing waves on the surface of water. It is so much a part of our mental model of the natural world that we go back to this phenomenon time and again in order to build our intuition of other forms of wave propagation in nature, even when the analogy is more challenging to our imagination than the other forms are in themselves. The salient features of water waves can be qualitatively grasped by everyone, from kids playing in a pool to seamen on a ship, and have long stimulated curiosity in anyone who watches the waves. It is therefore perhaps ironic that our quantitative understanding of water waves is still limited, and that our knowledge in that area is behind that of other less familiar types of wave motion. Of course, there are good reasons for this limitation. Of all the waves in nature, water waves possess one of the widest ranges of variation in their behavior, from the small amplitude long waves at the surface of a quiet pond to the crashing turbulent surf breaking on a beach. Such a wide range of scale challenges our mathematical models, particularly when it combines with another momentous aspect of water waves – nonlinearity.

Just as our first experience with wave motion came from water waves, so did perhaps our first appreciation of the role played by nonlinearity in physical models. Observing a beach wave approaching a shore, one is struck by its emerging as a single identifiable entity out of an almost disorganized state of the sea surface far from the shore. The wave seems to gain coherence as it approaches the beach and grows taller. Why is this coherence lost when the wave steepens and breaks on the shore? What is the mechanism behind the delicate balance between height, width, and speed? Water waves, and fluid dynamics more generally, is one of the fields in mathematical physics where we were first forced to cope with nonlinearity, with no hope of discarding it while still retaining physically interesting solutions. We have learned how to incorporate nonlinearity's subtle ways as a creator and destroyer of stability in simple models of water wave motion. Why do these models work so well – or do they?

Fluid dynamics abounds with examples of nonlinear waves. Waves spontaneously appear in water flowing down an inclined plane, at interfaces between different density layers in deep bodies of water, on the surface of a tray of water on a vibrating table, and are manifest as well in the large scale meandering of atmospheric winds at mid-latitudes, and at the enormous fronts of expanding gases from a supernova explosion. Osborne and Burch [67] describe the effects of nonlinear ocean waves that have traveled unchanged for hundreds of miles on the surface of the Andaman Sea near northern Sumatra. Each wave is approximately 100 m wide and is separated from the other waves by about 10 km. These waves, generated by tidal forces, move about 2 m/s and are seen as small (about 1 m) surface breaking waves. They are the tiny surface signatures of huge internal waves that exist at the interface between the stratified thermal and salinity layers in the ocean. The beautiful mathematics developed to understand these ocean waves can be used in an attempt to describe the stability of vortex structures. One spectacular example is the giant red spot on Jupiter, a 40 000 km wide storm which has changed little since it was first observed in the early 17th century. The underlying similarity of all these phenomena is the nonlinear balance of forces between the dissipation, dispersion, and fluid convection forces. These are the principles that unify nonlinear waves in fluid systems of all scales, from millimeters to 10^4 kilometers and beyond.

We will give a selective review of some recent advances in the study of nonlinear dispersive waves, in particular their stable manifestations as solitary waves, primarily in water. The remarkable creation, stability, and interactive properties of these waves as they encounter external forcing, like variable depth, have been extensively studied analytically, computationally, and experimentally. The broad agreement between these three approaches is amazing, and they have provided a useful beacon when they do not agree, signaling where to look for new understanding.

2.1. Nonlinear waves in fluids

The first published observation of solitary waves was made by John Scott Russel along the banks of a shallow canal in Scotland. He reported his observation of a wave propagating with no appreciable change in form, as well as the results of decade-long experimental study, in his famous paper "Report on waves" of 1844 [73]. His work having been largely ignored for quite some time after its publication must have generated some sense of guilt in the scientific community, and it has become customary, as we have now done, to begin a review on the subject of nonlinear water waves with a reference to his work. To this, we would like to add a reference to another pioneer in the field, Joseph Boussinesq. His publication [8] of what we now call the Korteweg–de Vries (KdV) equation,

$$u_t + u_x + uu_x + u_{xxx} = 0, \tag{1}$$

preceded by more than two decades the derivation by Korteweg and de Vries, published in 1895 [48]. The KdV equation (1) is written in the usual nondimensional form, in which all the physical parameters are scaled into the definition of space x, time t and velocity of water $u(x,t)$. Not only was Boussinesq the first to derive the model, but he also carried out some nontrivial analysis and found the first members of an infinite hierarchy of conservation laws [60]. Boussinesq must also be credited for the first theoretical evidence of solutions that resembled Russel's solitary wave. His rationale for how such solutions can come about is the one we still favor today: two "destabilizing" effects, linear dispersion (represented by the u_{xxx} term in the KdV equation), and nonlinear convection (represented by the uu_x term in the KdV equation) work against each other and exactly balance to result in a stable solution. So it is perhaps another example of history's injustices that the model should be known as KdV, although Korteweg and de Vries do deserve the credit for having isolated Eq. (1) among Boussinesq's many alternative models and for calling attention to its solitary wave solution in their paper [48].

The KdV equation had to wait almost another century before it was finally given its place as a paradigm for all nonlinear wave models by the work of Zabusky and Kruskal in the mid-1960s [82]. At that time computers were making their first appearance in general scientific research, and effectively turning the branch of mathematics related to evolution equations into an experimental science. In numerical experiments following those of the seminal work of Fermi et al. [19], Zabusky and Kruskal discovered that an initial sine wave would break into a train of stable solitary waves under the flow of the KdV equation. They found that when this nonlinear solitary wave collided with other solitary waves it emerged unchanged, except for a phase shift – a remarkable property, and they gave the name "soliton" to these particle-like solitary waves.

The simulations of Zabusky and Kruskal motivated the mathematical community to search for new analytic tools to explain the phenomena. The enticing behavior of the KdV solitons slowly gave birth to some of the most significant advances in applied mathematics during the 20 century. In an ingenious tour de force, Gardner et al. [23] put together in 1967 the elements of what is now known as the "inverse spectral transform" (IST), a sequence of essentially linear operations that allows the construction of the solution of the KdV equation. The IST identifies the soliton (particle-like) components of the solution as nonlinear modes (a generalization of the Fourier modes for linear PDEs) which coexist with modes that are purely dispersive (radiation-like).

Remarkably, the KdV equation can describe the main features of a range of fluid waves from the small narrow channel where Russel chased the soliton to the huge internal waves flowing in the Andaman Sea. Of course, it would be unfair to ask such a simple model like the KdV equation to stretch beyond the level of a qualitative understanding over such a wide range of physical situations, though quantitative agreement between experiments and KdV theory does exist and can often be amazingly good. Over the years, many more refined models have been proposed. Starting with the one and two equation models proposed in the original works of Boussinesq, we also note here the alternative model proposed by Benjamin et al. [6], which points to the idea of "regularization" of a dispersion relation and its consequences for improved numerical and physical behavior. Unfortunately, this model does not lend itself to IST as we know it, but an integrable generalization has been proposed [13], with interesting solution properties.

Despite its success at providing a framework for the Andaman Sea internal waves, the KdV can fail to be the appropriate asymptotic model in the presence of very deep layers of almost homogeneous fluid. For such a case the correct asymptotic model was found by Benjamin in 1967 [5], a nonlocal version of the KdV equation in that one of the derivatives of the dispersion term u_{xxx} is replaced by a Hilbert transform. This model, commonly referred to as the Benjamin–Ono (BO) model, is also completely integrable by the IST, but the application of this method is far more difficult than its counterpart for the usual KdV model. The IST theory for the BO equation has been recently used for evaluating the effects of perturbations. A sequence of posters at NWSPS by Kaup, Lakoba, and Matsuno presented the latest advances on this subject. We also note that the subject of near integrable equations and in particular of perturbation theories based on IST have been given new life by the recent advances in fiber-optic communications (Section 3).

The BO model is actually the limit, as the depth of one of the fluid layers goes to infinity, of another nonlocal model known as the intermediate long-wave (ILW) equation, which can also be solved by IST. In his lecture at NWSPS, Bona presented new advances on the global existence theory of coupled ILW equations, a system that is appropriate for waves at the free surfaces of a shallow layer of water sandwiched between two deep layers of different densities.

An effort that lately has received some new impetus is that of investigating the consequences of varying the relative balance of dispersion and nonlinearity that goes into making a solitary wave. Although the experimental observations of waves that are long with respect to the total water depth confirm the KdV balance (of the wave amplitude scaling like the square of the wavelength), for internal waves in the presence of a deep fluid layer the Benjamin model has proved to be inadequate (Koop and Butler [47]). The main reason for this inadequecy seems to be the strong nonlinearity, as expressed by a suitable amplitude parameter, that internal waves can achieve in deep water, and make an asymptotic model based on the smallness of this amplitude parameter inappropriate. Inclusion of higher-order terms in the long-wave small-amplitude asymptotic expansion that leads to BO offers no relief: the expansion itself becomes disordered for sufficiently long waves (Matsuno [55]).

Recently, a class of models capable of handling large amplitude waves has been proposed, and the comparison with existing experimental data [15] is very encouraging. These models can be viewed as a modification of the classical Boussinesq systems for homogeneous fluid in the same spirit as the BO equation modifies KdV. It is an open question whether any of these new equations possess some of the structures that make KdV and BO such interesting mathematical objects.

In this issue of Physica D, Wu [81] has made the search for models that account for various degrees of importance of nonlinear and dispersive effects more systematic. His starting point is a reformulation of the Euler equations for an incompressible, inviscid, and irrotational fluid into two (exact) equations. The first equation is the continuity equation averaged over the water depth, and second equation is the horizontal component of the momentum equation at the free surface. The equations have three unknowns: horizontal velocity at the free surface (in two dimensions), the mean horizontal velocity averaged over the water depth and the water surface elevation. By closing the system with a third approximate equation that relates the velocity fields, most previously known models can be recovered and extended.

Because of the generality of the formalism, the number of spatial dimensions in these models requires no restriction. If, however, the variation of the wave pattern in one direction is assumed to be weak, the Kadomtsev–Petviashvili (KP) equation [38] can be derived as a (weakly) two-dimensional generalization of the KdV equation. Like the KdV equation, the KP equation is a completely integrable Hamiltonian system and admits a large family of periodic and quasiperiodic solutions. The periodic cnoidal wave solutions of the KdV equation correspond to the one-phase KP solutions. Multi-phase (quasiperiodic) solutions expressed in terms of Riemann theta functions are also known. The two-phase KP solutions are the simplest genuinely two-dimensional ones and form permanent spatial patterns of crossing traveling waves. These two-phase solutions have been observed on ocean bays and have been produced in laboratory experiments. The agreement with the KP model is sometimes as spectacular as for the KdV equation. Unlike KdV however, the KP equation does not yet have an IST solution of the initial-value problem, a severe limitation for practical applications. Some progress in the direction of a solution to the initial-value problem is established for quasiperiodic initial data in this special issue by Deconinck and Segur [17].

As remarked in Section 1, the IST can be viewed as a sort of nonlinear Fourier transform. Osborne et al. [68] push this analogy to its ultimate consequences – time series analysis by the periodic IST. When shallow water surface data from the Adriatic Sea are analyzed with this new tool, a very appealing physical picture emerges: oceanic wave trains in shallow water appear as a composition of a "soup of particles" (the nonlinear cnoidal waves and solitons) living in a sea of "intermediary particles". The latter are responsible for the interactions among the primary stable particles (nonlinear modes) and in particular determine the resulting phase shifts. Besides its theoretical (high energy physics like) appeal, this picture is relevant in the design of offshore structures, like drilling platforms, where it is necessary to account for the type of wave forcing that the structure is likely to experience. Additionally, in his talk at NWSPS Osborne also presented some new experimental and theoretical advances for the problem of waves running up an inclined bottom topography and reaching the limit of sharp crests.

Some of the physically important perturbations of the above models come from wave interaction with bottom topography. When the bottom is slowly varying, the long wave KdV-like equations can be augmented with variable coefficients to account for the change in water depth. Milewski [57] derives appropriate asymptotic models and shows that, for unidirectional changes of the topography, two-dimensional solutions that vary slowly along the direction of constant depth can be described within the formalism of one-dimensional models.

The KdV equation arises in situations other than gravity waves. For example, Ludu and Draayer recently used the KdV equation to model the nonlinear oscillations on the surface of a drop of liquid [52]. They apply the model to alpha and cluster formation in heavy nuclei and demonstrate that the predictions agrees extremely well with experimental data.

The KdV equation has also been used by Malfliet and Ndayirinde [53] to model solitary waves (pressure pulses) of an incompressible fluid confined in along thin viscoelastic circular cylinder. They derive many of the relevant quantities, such as the wave speeds, by a direct perturbation technique. They investigate the influence of the different parameter values and compare the results to experimental data.

In an investigation related to the subject of internal waves in fluids of variable density, Goez [24] derived a new weakly nonlinear wave equation to follow the suspension of two-phase flows in fluidized beds (e.g., gas bubbles finely distributed in a liquid). Uniform states in fluidized beds can be unstable and bifurcate subcritically into a branch of periodic traveling waves. He derived a KdV-type equation governing long-wave dynamics by using the Froude number as a small parameter that measures both the size of the perturbation terms and the strength of the instability. The traveling-wave solutions of the model turn out to be unstable for very dense or very dilute fluid beds, sometimes with finite time blow up, although in the full equations these instabilities eventually saturate. Thus the model can only be used to capture the onset of instability.

2.2. *Waves and patterns near instabilities in fluid flows*

The onset of instabilities often occurs in the form of traveling wave patterns. Porta and Surko [70] have examined a type of convection instability that occurs in mixtures of water and ethanol far from equilibrium. Their experimental setup includes a cylindrical convection cell with a large aspect ratio. Initially, they observe small domains of large-amplitude traveling-wave convection patterns, separated by regions of "cross-roll" instability. These patterns evolve to a globally rotating state where large regions of convection rolls (oriented perpendicular to the boundaries) travel around the cell and are separated by small regions of defects. The analysis of Porta and Surko focuses on identifying parameters in the complex patterns they obtain experimentally to establish quantitative means of comparison with analytical and numerical models. They point out that quantitative information on the evolution cannot be obtained by a single technique. For instance, the spatiotemporal Fourier transform can be used to compare the temporal evolution of the system with its spatial behavior, but detailed information on the latter can be gathered only by focusing on the evolution of a small subdomain of a pattern and by making the assumption that a pattern is formed by a single (deformed) wave component. The investigation of Porta and Surko is particularly significant because the question of how to extract quantitative information from experimental observations is not confined to their particular experiment, but extends to the study of many nonequilibrium systems, such as chemical waves and nonlinear optical laser systems, where complex patterns are likely to arise.

Wave patterns commonly arise in vertically driven fluid layers. Because of the competition of gravity/vertical acceleration forces and capillary forces, beautiful wave patterns appear at the free surface of the fluid in a phenomenon known as Faraday instability when parameters exceed certain thresholds that depend on the frequency and amplitude of the driving. When nonlinearity is important, these patterns can have a rich underlying mathematical structure based on their symmetries, and a recent theoretical study by Silber and Proctor [75] presents many generic possibilities. Kudrolli et al. [49] offer an experimental study of some of these patterns in fluid driven by a two-frequency vibrating table. They describe the patterns they observe as "superlattices", because they can be thought of as a composition of two different hexagonal sublattices interacting with each other. This observation further extends the already vast variety of symmetric patterns that nonlinear surface waves are known to assume.

Symmetric patterns are but one of the fascinating effects of instabilities in parametrically driven fluids. Increasing the amplitude of the forcing excites a stronger nonlinear response of the fluid, and the Faraday waves can focus their energy into a violent erupting fluid spike, or jet. When the excitation of these frequency-locked periodic wave states exceeds a threshold, the modulated waves break into an aperiodic state with erupting spikes, ejecting droplets and entraining air. In this special issue, Hogrefe et al. [35] describe their experimental observations of such phenomena. They also introduce a low-dimensional model of the fluid dynamics and singularity formation based on return maps of the wave height. Using this analytical tool, they are able to accurately predict the temporal dynamics and threshold for the critical focusing of the energy in individual spiking events observed in their experiments. The spiking event signals the formation of a singularity at the fluid's free surface. The singularity observed in the experiments by Hogrefe et al. not only appears as a loss of smoothness of this surface but also shows a tendency to a blow up in amplitude (height). Of course, in reality infinite amplitudes cannot be achieved, and an upper bound is provided by a Rayleigh instability cut-off, which breaks the thin neck of the jet into droplets.

Another example of the underlying unity of nonlinear science in fluids appears in the paper by Forest and Wang [21] who studied the Rayleigh instability of cylindrical jets of a liquid polymer. This instability is due to the competition of surface tension forces associated with different radii of curvature of the jet, and is amplified by the exponential growth of the amplitude of long axisymmetric waves at the jet's free surface. Controlling the onset of this instability is important in industrial applications, as in the case of liquid crystalline polymers (LCPs) which are used for everything from sports coats to sports cars. The molecular-scale microstructure created by spinning the fibers greatly affects their performance. The internal orientation of the slender fibers in the spinning process can be modeled by a wave equation. The stability of the spinning process is governed by the Rayleigh capillary instability of the coupled hydrodynamic and orientation effects. The strong coupling of the internal orientation of the microstructure to the hydrodynamics in the filaments significantly alters the stability boundary from the classical inviscid analysis. Forest and Wang compute the linearized stability boundary in terms of the LCP effects such as kinetic energy, relaxation, anisotropic drag, and the upstream degree of orientation. Their analysis confirms that anisotropic oriented filaments are less susceptible to capillary instabilities than are similar filaments of homogeneous, isotropic fluids.

Perhaps the most classical example of wave patterns generated by instabilities in fluids is that of Rayleigh–Bénard (RB) convection. When convection takes place in a rotating environment, several new effects, which are due primarily to the Coriolis force, come into play (see [14]). This situation is especially interesting because it occurs naturally in planetary atmospheres, in oceans, and in solar plasmas, and in experiments of thermal convection with rotation about a vertical axis in a controlled laboratory environment can greatly help the modeling of these large-scale fluid motions. Vorobieff and Ecke report some new observations of the turbulent velocity field in the unstable thermal boundary layer at the top (cold) plate of a rotating RB cell [80]. By quantifying the velocity field through the scanning particle image velocimetry measurements, they have been able to confirm earlier theoretical predictions by Julien et al. [37] about the combined action of viscosity and rotation (Ekman pumping) in the boundary layer. The findings of Vorobieff and Ecke point to the mechanism of Ekman pumping as a possible explanation of the heat transport enhancement observed in turbulent rotating RB convection.

Instability thresholds are usually expressed in terms of dimensionless numbers. They appear everywhere in fluid dynamics, and provide a unifying viewpoint for fluid flows that can occur over a range of several orders of magnitude in relevant scales. In the RB convection experiments previously mentioned, the instability number, representing a measure of the competition between viscous and buoyancy forces, is called the Rayleigh number. The paper by Doering and Wang [18] deals with what could be arguably considered the most famous nondimensional number in fluid mechanics, the Reynolds number (*Re*). This number offers a measure of the competition between inertial and viscous forces. It is in terms of this number that the threshold between ordered laminar flow and disordered turbulent motion is expressed. Turbulent motion may well be considered as the ultimate complex behavior exhibited by nonlinear viscous fluids. From a dynamical systems viewpoint, under certain conditions, the long time dynamics of these fluids can be governed by a *finite* (albeit possibly very large) numbers of degrees of freedom [16]. The infinite-dimensional phase space of the fluid can "flatten" as time progresses and can concentrate onto a finite-dimensional attractor. While this picture presents an obvious conceptual appeal for studying turbulence, there seems to be no general recipe, given a certain flow, for determining the exact number of degrees of freedom, or at least a sharp upper bound for it. The paper by Doering and Wang examines the far end (away from laminar instability) of the Reynolds's number spectrum, as $Re \to \infty$ in two-dimensional incompressible shear flows modeled by the Navier–Stokes equations. Doering and Wang establish rigorous estimates of the asymptotic dependence on *Re* of the attractor dimension, considerably sharpening the previously available results. For these turbulent channel flows, the estimates of Doering and Wang in turn rigorously define, from first principles, the smallest spatial scale, a quantity that is of great importance for adjusting the resolution of numerical and experimental studies of such flows.

3. Optics

The creation of an information superhighway based on-terabit-per-second optical-fiber transmission systems presents an exciting opportunity for the applied mathematics and optical communications communities. With the development of all-optical, erbium-doped fiber amplifiers in the 1980s, all-optical networks became possible, and immediately the long time propagation properties of pulses in fibers became relevant to achieving higher performance. Indeed, nonlinear fiber optics is one of the best meeting places between some of the most interesting techniques of applied mathematics and practical engineering.

In this section, brief descriptions of many of the talks and posters given at NWSPS which are directly related to optics are included, in addition to descriptions of all the optics-related papers in this volume. The former are included because these materials aptly illustrate the state of the art in this focused and quickly advancing field.

Under the assumption of a single optical polarization, the equation governing the electric field envelope in a single-mode optical fiber turns out to be the perturbed Nonlinear Schrödinger (NLS) equation,

$$\frac{\partial q}{\partial Z} = d(Z)\frac{\mathrm{i}}{2}\frac{\partial^2 q}{\partial T^2} + \mathrm{i}|q|^2 q - F\left(Z, T, q, \frac{\partial q}{\partial T}, \ldots\right), \tag{2}$$

which is written here in nondimensional "standard soliton units", as they are commonly known. The coordinate T is the nondimensional time in the retarded frame associated with the group velocity of wavepackets at a particular optical carrier frequency. The nondimensional coordinate Z is the longitudinal spatial coordinate along the fiber. The funtion $d(Z)$ takes into account variations in group velocity dispersion along the fiber, while the perturbation F takes into account losses, higher-order linear and nonlinear corrections, periodic amplification, and any other optical processing imposed on pulses. The NLS equation was first derived for optical fibers by Hasegawa and Tappert [33] in 1973. For a detailed derivation of this equation using multiple scales techniques in the context of optical fibers see, for instance, the book by Newell and Maloney [65].

If higher-order effects, losses, and variations in dispersion are neglected, i.e., if $F = 0$ and $d(Z)$ is a constant, then in both the defocusing ($d < 0$) and focusing ($d > 0$) cases the unperturbed NLS equation is a completely integrable Hamiltonian system, on the real line and also under periodic boundary conditions, via the inverse spectral method [20,71,83,84].

On the real line the NLS supports N-soliton solutions, the 1-soliton solution being given by

$$q(Z,T) = \eta \operatorname{sech}(\eta[T - \kappa Z - T_0]) \times \exp\left\{-\mathrm{i}\kappa T + \frac{\mathrm{i}}{2}(\eta^2 - \kappa^2)Z + \mathrm{i}\sigma_0\right\},$$

where η, T_0, κ, and σ_0 are the (real) soliton parameters. Various properties of soliton solutions and the way these depend on physical parameters are crucially important for fiber optics applications. For example, the pulse-width and amplitude parameters are both identified with η, so that these properties are slaved. This slaving is a direct consequence of the balance between nonlinearity and dispersion that the soliton solution embodies. Subsequently, changes in the energy or amplitude of pulses due to perturbations tend to produce changes in pulse-width as well, leading to a complicated interaction between nonlinearity, losses, and dispersion. This slaving also means that smaller pulse-widths require higher peak amplitudes, with the consequence that higher-order nonlinear effects become significant when one tries to decrease pulse-width to increase the bit rate. These nonlinear effects, as well as higher-order linear effects, typically become significant for pulse-widths on the order of a picosecond or less, corresponding to bit rates of approximately 100 gigabits per second or larger.

Over the years, there have been many significant contributions to the development of an NLS soliton perturbation theory to describe the effects of perturbations. Among these are the fundamental contributions of Kaup [40–42], Keener and McLaughlin [43], Kodama and Ablowitz [46], Karpman and Solev'ev [39], and Kodama and Hasegawa [31,32]. Biswas gave a poster at NWSPS presenting an advanced multiscale perturbation theory of the NLS equation, which obtains a consistent slow dynamics that was not obtained with the usual soliton-ansatz of previous theories.

The solitons of the unperturbed NLS equation collide elastically; that is, they completely regain their initial form after collisions with only a change in phase. They are also the only stable pulse solutions of the NLS equation, in the sense that small perturbations do not grow asymptotically in time in an appropriate norm. These properties suggest that, in an ideal world, they would be the ideal carriers of information in fiber-optic links. This elastic property of soliton interaction makes the soliton particularly appealing for use in the so called Wavelength Division Multiplexing (WDM) technique. This techniques consists of transmitting data streams in multiple frequency channels within the same optical fiber. WDM is generally preferable to using a single channel of extremely short pulses, called Time Domain Multiplexing (TDM), to take advantage of the enormous bandwidth available in optical fibers with the present technology. The reason is that the amplitude-pulse-width slaving by the parameter η quickly leads to very destructive higher order effects in broad-band TDM systems. Unfortunately, different frequencies in the WDM technique mean different (group) velocities, and so pulses in the data streams would necessarily undergo multiple collisions. These interactions among channels would eventually corrupt the structure of the bit patterns. The elasticity of the collision for solitons provides an elegant solution to this problem.

The feasibility of soliton based WDM systems has been recently demonstrated in experiments that have achieved bit rates of terabits per second over short distances, and hundreds of gigabits per second over thousands of kilometers. Of course, in actual experiments the unperturbed NLS equation is only a rough approximation, and the perturbations that one needs to add to NLS destroy the elasticity of soliton collisions. In order to achieve the high bit rates above, investigators [54,61] had to devise clever ways of controlling the effects of perturbations to make the collisions as elastic as possible.

Arguably the most important effect limiting the elasticity of soliton collisions is due to the persistence of sideband frequencies generated during a two soliton collision in presence of perturbations. These residual sideband radiative modes interact with the solitons gradually perturbing their phases and hence their positions within their time slots, leading to timing "jitter", and hence communication-bit-error rate.

At NWSPS, Biondini presented a poster on his fundamental work with Ablowitz on four-wave mixing [1]. By deriving analytical expressions for the four-wave mixing terms in lossless fibers using an asymptotic expansion of the N-soliton solution of the NLS equation, this work showed that the four-wave mixing increases by an order of magnitude when losses and periodic amplification are present. Moreover, a resonance condition between the soliton frequency and the amplifier distance was derived that correctly predicts all the relevant features of the four-wave mixing in real fibers.

Research on ways to increase the capacity of optical communication systems, regardless of the particular format, is naturally very active. One the most promising techniques in this direction is the so-called dispersion management approach, tested in soliton-transmission system in 1995 by Suzuki et al. [76]. In this technique the group velocity dispersion of the fiber, represented by the coefficient $d(Z)$ in Eq. (2), is made to switch periodically along the fiber between positive and negative values, so that the path averaged value of $d(Z)$ remains small and positive. In soliton systems, this allows to decrease the effective width of the pulse.

In this special issue of *Physica D*, Kodama presents a detailed theory of these pulses [45] in a single channel. For weak variations of the dispersion $d(Z)$, he employs the so-called guiding center, or normal form theory, based on the Lie transform, which he and Hasegawa had previously introduced to the fiber-optic community with great impact [31,32]. In this paper Kodama also shows that for large variations of dispersion the leading order equation is the NLS with a quadratic potential, a result previously obtained by Hasegawa and Kumar based on the Talanov transformation and reported in Hasegawa's NWSPS paper [30]. In this paper and NWSPS talk, Hasegawa has analyzed a new type of nonlinear stationary optical pulse, or quasi-soliton, generated by inserting an optical "chirp" with a programmed time-dependent frequency into a fiber that has been carefully doped to create an inhomogeneous disperion coefficient. The quasi-soliton is the solution of the NLS equation augmented with quadratic potential, and has a shape between Gaussian and hyperbolic secant. Being effectively narrower, this pulse is potentially better suited for ultrahigh-speed transmission lines in specially designed fibers.

In another interesting application of the Lie transform technique to the slowly varying spectrum of a pulse in dispersion managed system, Gabitov in his talk at NSWPS obtained a new equation which models slow nonlinear dynamics of the pulse spectrum. Using this new equation he proposed a dispersion profile that effectively reduces accumulation of four-wave mixing sidebands in WDM systems, thus decreasing the bit-error rate in such systems.

Finally, two other posters presented at NWSPS dealt with the properties of the dispersion managed pulses. The poster by Turitsyn gave a theory of these pulses and examined their stability. A poster by Cruz-Pacheco derived traveling-wave solutions for dispersion-managed lines by using a perturbative approach, and showed that fast jitter occurs when an initial frequency correction is included.

A major source of timing jitter which is present even in single channel (TDM) systems is the phase shift that pulses acquire from the addition of spontaneous emission noise from the optical amplifiers used to compensate losses. This specific forms of timing jitter is called the Gordon–Haus effect [26]. A very effective method for reducing the Gordan–Haus effect and timing jitter in general, called sliding-frequency guiding filters, was proposed in 1992 by Mollenauer et al. [62]. In this technique, the pulses are filtered periodically with filters whose center frequencies slowly shift along the fiber. Due to their intrinsically nonlinear nature, the solitons can adjust their spectra to follow the filters, provided that some additional gain is provided to offset the dissipative effects of the filters. Noise, which has a much smaller intensity than that of the solitons, evolves linearly, and therefore cannot follow the filters and is hence reduced.

In an NWSPS poster, Horne showed that the error rate from collision-induced timing jitter WDM soliton systems with sliding-frequency filters only grows linearly with distance, a result that was previously known only through numerical simulations.

Ultimately, the best approaches may involve combinations of several techniques, including dispersion management, sliding-frequency filters, and dispersion tapered fibers. The latter technique is one whereby the broadening in pulsewidth that is usually accompanied by loss in energy is eliminated by exponentially, or approximately exponentially, decreasing the group velocity dispersion, i.e., $d(Z)$ in Eq. (2), with Z along the fiber. In talks given at NWSPS, Mamyshev [54] and Mollenauer [61] discussed the various pros and cons of such combinations. For example, they described how the WDM approach using multiplexed dispersion managed solitons in a transmission line with sliding-frequency filters has the potential for transmitting up to 10 or 20 gigabits per second in each channel over essentially infinite distances. With several tens of channels, yielding total bit-rates of 100 gigabits per second and up, these results are a very good example of the state of the art.

Sliding-frequency filters may also be used for other purposes. For example, a poster was given by Burtsev and Camassa at NWSPS, described the use and optimization of sliding filters to convert pulses in the non-return-to-zero (NRZ) format to the soliton format (return-to-zero, or RZ format). In this work, a massive application of the inverse scattering transform was implemented numerically to observe the soliton content of pulses.

The reduction of timing jitter may also be approached by all-optical regeneration of the data stream. In this special issue, Niculae and Kath [66] present analytical and numerical studies of an all-optical clock recovery system which uses a data stream to mode-lock a fiber laser through cross-phase modulation to reduce the timing jitter. By transmitting data and laser pulses simultaneously in a fiber, the cross-phase modulation induces mode locking. Using soliton perturbation theory, they analyze the situation when the laser pulses are close to optical solitons and show that the temporal walk-off between the signals has an important role in reducing the timing jitter.

Other current research questions focus on the interactions between orthogonal polarizations in the fiber. About 10 years ago, Menyuk [56] showed in numerical simulations that at high power two polarizations can interact and strongly couple two initial pulses with opposite polarization into a single pulse with both polarizations. In an NWSPS poster, Lakoba presented a perturbation theory for the soliton of the Manakov equations and applied the theory to the problem of soliton propagation in randomly birefringent fibers. Lakoba analyzed the slow evolution of the soliton and the emitted radiation, and obtained results that support earlier numerical simulations and a simple intuitive description of the soliton dynamics viewed as quasiparticles.

Polarization effects also play an important role in fiber-optic devices such as couplers. In this special issue, Valkering et al. [79] employ a coupled NLS equation to model optical couplers. They show that switching occurs when an incoming symmetric soliton is unstable in the coupler and that a stable asymmetric soliton with the same energy as the incoming pulse exists concurrently. They approximate this mechanism in a simple three-dimensional phase space analysis, which suggests that the energy transfer between the two channels can accurately predict the sharp transition between the switching and nonswitching behavior.

The present state-of-the art in optical communications is NRZ systems. These were analyzed in detail by Kodama in his NSWPS talk. In particular, he examined the case of zero dispersion limit for such systems, in agreement with results of the general theory developed by McLaughlin and collaborators in the early eighties.

Research on solitons in dispersive media with a second-harmonic-generating quadratic nonlinearity has been thus far focussed on stationary spatial solitons in one-dimensional media or two-dimensional media with cylindrical symmetry. Unlike a cubic nonlinearity, the quadratic nonlinearity does not give rise to the wave collapse in two- and three-dimensional media and potentially could produce stable "light bullets", i.e., fully localized spatiotemporal solitons, in bulk dispersive quadratic media. Musslimani and Malomed [63] demonstrate that continuous wave solutions to the second harmonic generation equations in a multidimensional quadratic nonlinear dispersive medium are always modulationally unstable. They also consider the modulational stability in a more general three-wave system.

More general and fundamental aspects of nonlinear wave mixing are also of interest to the nonlinear optics community. For example, in this issue Alber et al. [4] derive the Hamiltonian Lie–Poisson structures of the so-called three-wave equations associated with the Lie algebras of SU(3) and SU(2) and show that they are compatible. In a poster presented at NWSPS, Biondini and Ablowitz used an asymptotic expansion of the N-soliton solution of the NLS equation to derive the asymptotic wave equations governing the envelope of a light pulse in a nonresonant quadratic material. They showed that under proper conditions, the system leads to vector NLS-type equations that appear to be generalizations of the Davey–Stewartson equations.

Also in this issue, Miller and Akhmediev [58] investigate multisoliton interactions in planar waveguided optics modeled by vector NLS equations. Their analysis by separation of variables constructs an exact solution of a linear Schrödinger equation for a class of potential functions that directly relate to the multisoliton collisions.

4. Materials science

Dynamics on discrete lattices is an old area of nonlinear science (e.g. the Fermi–Pasta–Ulam experiment described below) that has recently seen an explosion of new interest, due primarily to the relevance of these models to problems in material science, solid-state physics, chemistry and biology. Unfortunately, the discrete lattice ordinary differential equations that arise so naturally in these systems are not as submissive to mathematical analysis as their continuous PDE counterparts.

In the early 1950s Enrico Fermi, John Pasta, and Stan Ulam conjectured that the energy in a one-dimensional lattice of masses connected by nonlinear springs, as modeled by the now famous FPU system of ODEs

$$\frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2} = (u_{i+1} - 2u_i + u_{i-1}) + \epsilon[(u_{i+1} - u_i)^2 - (u_i - u_{i-1})^2], \tag{3}$$

would thermalize, that is, for any initial condition the nonlinearities would cause the energy to cascade and distribute evenly among all the accessible linear modes of the system [19]. This conjecture was based on the post-Poincaré but pre-KAM intuition that the phase space is either filled with tori when the system is integrable (the case here when $\epsilon = 0$) or else nonintegrable with a homogeneous sea of ergodic trajectories (the case when $\epsilon \neq 0$) and associated stochastic dynamics. (Fermi published a proof attempt of this as early as 1923.) In what is now recognized as a cornerstone in the field of computer simulations of nonlinear systems, they solved the differential equations numerically on the MANIAC-I computer at Los Alamos. Numerical simulation was still in its infancy: each iteration (some simulations had 80 000) required refeeding the stack of punchcards. To their great surprise they found that instead of the expected gradual, continuous flow of energy from the first mode to the higher modes, the energy remained bound in the lowest few modes, cycling about in a quasiperiodic fashion. The solution again and again continued to almost return to its initial state.

Initial efforts to resolve this mystery basically took two different approaches: analysis of the discrete problem using perturbation methods and analysis of associated continuum equations extracted in the limit of small ϵ (the measure of the strength of the nonlinear interaction between neighboring particles). It was this latter route, first taken by Zabuski and later joined by Kruskal, which lead to the KdV equation, and from there to the discovery of solitons. However, as successful and rich as the study of associated continuum limits has turned out to be, there are many instances where the numerics or physics dictate that one cannot neglect the discrete nature of the problem. Firstly, extracting an appropriate continuum limit can be difficult, since discreteness can preserve the integrability that is often lost in the continuum limit. In the Toda or sine-Gordon lattices [77], the lattice has exact, localized oscillatory modes that are unstable in the simplest second-order

PDE approximation. That which is a discrete stable soliton for the lattice can form a discontinuity in the second-order continuum limit of (3). Secondly, the interaction between discreteness and nonlinearity is the reason that breather solutions are quite robust in discrete systems and so important to the understanding of the novel properties of many materials.

Techniques that are highly developed for continuum systems, such as the IST, have been generalized very little to discrete systems, except for the Toda and Ablowitz–Ladik models [2]. To address this gap, Peryard describes in his article [69] how to use the techniques of linear stability and analysis of wave propagation and how resonances can be used to better understand these discrete dynamics.

In his talk, Sievers described how, in some cases, the discreteness and lattice anharmonicity are precisely what is needed to create localized packets of vibrational energy that propagate as stable intrinsic localized modes on an anharmonic crystal. In addition to the usual kinetic and potential energy of the ac vibration in these modes, the localized distortion also holds energy. That is, the crystal structure is important to the stability and existence of these pulses, and they may be more likely found in zinc blend than fcc lattices. In computer simulations, Sievers found that the gap between the optical and acoustic branches of the plane wave phonon spectrum in pure diatomic crystals (with realistic potentials) is the most likely place to find these intrinsic localized gap modes. In an NWSPS poster, Kiselev and Sievers presented a one-dimensional anharmonic lattice model of a diatomic crystal. Here the intrinsic gap mode between the optic and acoustic branches allows for a localized vibration above the top of the plane wave spectrum.

At the NWSPS conference, Lai and Sievers described using direct MD simulations to check the stability of the intrinsic localized spin-wave modes in classical Heisenberg antiferromagnetic chains created by the nonlinear properties of the discrete lattice. They identified and studied the parameter space where localized spin-wave gap modes occur in easy-axis antiferromagnetic chains and where localized spin-wave resonances occur in easy-plane antiferromagnetic chains. They also investigated the modulational instability of extended nonlinear spin waves in antiferromagnetic chains using linear stability analysis and MD simulations. Their simulations verify the analytical predictions for short time scales. However, by the time the instability is fully developed, the linear stability analysis fails and the modulated spin waves can become chaotic.

Göktas and Hereman [25] recently implemented an algorithm to compute polynomial conserved densities of polynomial nonlinear lattices. The approach can generate the explicit form of conserved densities of differential-difference equations. Usually, the first few conservation laws have a physical meaning, such as the conservation of momentum and energy. The higher-order conservation laws can aid the study of both quantitative and qualitative properties of solutions.

The application of nonlinear self-focusing and pulse-shaping waves in low-loss, high-speed nonlinear transmission lines requires new analysis and novel materials. The usual linear approach of designing a resonant response is inadequate for high data rates. Extremely fast responses have been demonstrated in semiconductor-based distributed devices. However, the loss introduced by the nonlinear semiconductor elements limits their performance.

The discontinuities created by discrete components in communications, radar, and digital electronics are investigated in a perturbed Toda lattice model for low loss nonlinear transmission lines by Cai et al. [12]. They investigate incorporating nonlinear dielectric thin films of strontium–barium–titanate in coplanar waveguide devices to reduce the losses yet retain the nonlinear response. At the microwave frequencies of technological interest, these components have less loss than that of semiconductor components. Lai and coworkers have simulated and used perturbation theory to demonstrate how a nonlinear transmission line can shape an input pulse into a train of stable traveling solitons.

The microscopic mechanisms for mobility, friction, and lubrication processes is necessary for a better understanding of solid friction at a macroscopic level, as well as adhesion, contact formation, friction wear, lubrication, and fracture. The multiple-step dynamical phase transition from the locked to the running state of atoms in response to a dc external force is studied by Braun et al. [10], who used MD simulations in the underdamped limit of the generalized Frenkel–Kontorova (FK) model. They demonstrate how the hierarchy of depinning transition depends on the friction in a highly anisotropic quasi-one-dimensional rectangular potential and in an isotropic triangular system, where the interactions between neighboring channels play an important role in the dynamics.

Large-scale molecular dynamics (MD) simulations can give new insights into the dynamics of dry friction between two metallic interfaces. Hammerberg et al. [29] have used two-dimensional MD simulations using embedded-atom method potentials to study the nonlinear dynamics of slip at flat copper interfaces. Their studies include dislocation generation, dislocation motion both parallel and normal to the sliding interface, large plastic deformation, nucleation of microstructure, diffusive coarsening of microstructure, and material mixing associated with a velocity weakening of the tangential force at high relative velocities. Initially, when the flat sliding interface is dominated by dislocation motion parallel to the interface, they use a two-chain forced Frenkel–Kontorova model to reproduce some of the behavior of the large-scale MD simulations. In particular, this model exhibits four velocity regimes of steady-state flow and can be used to give insights into the nucleation of microstructure.

Rigid-ion, two-body potential MD simulations can be used to study the properties of intrinsic gap mode eigenvectors for crystal structures. By using an artificial dynamical simulated annealing technique of the Car–Parrinello-type crystals, Kiselev et al. [44] have simulated the anharmonic localization of lattice vibrations in a perfect three-dimensional diatomic ionic crystal. These eigenvectors consist of an ac vibrational component and a dc distortion of the lattice. Kiselev et al. also note that for the same crystal potential model, the intrinsic gap modes form more readily for the lower-symmetry zinc-blend structure than for the higher-symmetry fcc one.

Recently, the existence of anharmonic localization of lattice vibrations in a perfect three-dimensional diatomic ionic crystal has been established for the rigid-ion model by MD simulations. The technique was extended to three-dimensional anharmonic lattices with realistic potentials by calculating the eigenvectors of the strongly anharmonic localized modes using simulated annealing. The approach can also be used to investigate the stability of localized modes in other classical molecular models, such as the rigid ion model for ionic crystals.

The existence of breathers in discrete ϕ^4 field theory is relevant in solid-state contexts. In the continuum ϕ^4 theory, multiple-scale asymptotic perturbation theory arguments were the first to suggest the existence of ϕ^4 breathers. Later, it was discovered that there were terms beyond all orders in the perturbation expansion, that destroyed the putative breather, and only recently have rigorous proofs been given of the nonexistence of breathers in the ϕ^4 continuum theory. Motivated by the localized, nonlinear soliton excitations in the classical, nonintegrable ϕ^4 field theory and the oscillatory kink–antikink resonances, Campbell in his talk at NWSPS investigated the existence of spatially localized, time-periodic, nonlinearly stable breather solutions. He and his colleagues have provided a heuristic explanation of the stability (and instability) of kink–antikink interactions found in numerical simulations that show an intricate interweaving of stable and unstable breather solutions on finite discrete lattices.

In an NWSPS poster, Berman, Bulgakov, Campbell, Gubernatis, and Sadreev studied the quantum soliton wave function in mesoscopic ϕ^4 theory. They considered the phase transition of the ground state of the one-dimensional finite quantum discrete ϕ^4 model as a function of the strength of the quantum fluctuations. By comparing the phase diagram generated by variational methods with quantum Monte Carlo simulations, they showed that in the region of weak coupling, both the tunneling soliton wave function and the two-level approaches provide close approximations to the true results.

Generalized coherent states are defined as points of the factor spaces SU(2)/U(1). In the study of dynamics of the ϕ^6 lattice model in classical nonlinear field theories, Agueroa et al. [3] discuss the remarkable properties of the solitons in the generalized coherent states. Also, it has been shown that a certain curve in the "effective phase space" of the three-state quasispin model exhibits soliton solutions. By using a lattice with nearest-neighbor exchange interaction, remarkable properties (like the condensation of bubbles) occur for the soliton solutions during phase transitions.

The continuous wavelet transform is an effective tool for analyzing the frequency response of these discrete breather modes. Forinash and Lang [22] have investigated the conditions for the existence and stability of discrete breather states, which exist in the gaps outside of linear dispersion bands in anharmonic lattices. They apply a continuous wavelet transform to the numerical solutions and verify some of the predicted frequency behavior. The wavelet transforms can offer a significant improvement over Fourier transforms in engineering applications where it is desirable to have information about frequencies that change in time.

The n-kink solitary wave solutions for the parametrically forced sine-Gordon equation with a fast periodic mean-zero forcing represent a new class of solitary-wave solutions of the equation. In a poster at NWSPS, Zharnitsky, Mitkov, and Levi have applied this result to quasi-one-dimensional ferromagnets with an easy plane anisotropy in a rapidly oscillating magnetic field. In this case, the n-kink solution corresponds to the uniform "true" domain wall motion, since the magnetization directions on opposite sides of the wall are antiparallel. Using the normal form technique, Mitkov and Zharnitsky [59] have shown that the parametrically driven sine-Gordon equation with a mean-zero forcing is well approximated by the double sine-Gordon equation. Furthermore, the reduced equation possesses n-kink solutions, which are also observed numerically in the original system. They have applied the equation to model the domain wall dynamics in one-dimensional easy-plane ferromagnets where the existence of the n-kinks reflects the true domain structure in the presence of a high-frequency magnetic field.

In his talk at NWSPS, Grönbech-Jensen studied the dynamical properties of coherent modes and resonances in coupled sine-Gordon systems describing coupled transmission lines. New multiple-characteristic spatial and/or temporal scales arise for both linear and nonlinear modes in spatially distributed, coupled transmission lines. Furthermore, the interaction and phase-locking between nonlinear coherent modes affect the metastability and dynamics of inductively coupled long Josephson transmission lines (JTLs).

Vertical stacks of high-tellerium Josephson junctions are provided naturally by the intrinsic Josephson junctions in layered high-tellurium superconductors. These systems show strong inductive coupling that is due to the extremely thin superconducting electrodes. This coupling breaks the Lorentz-invariance of the sine-Gordon equation and thus gives rise to new phenomena. The coupled sine-Gordon systems can be used to model vertical stacks of Josephson junctions and, in long Josephson junctions, fluxons can be used as a model for solitons for these systems. Hechtfischer et al. [34] study the coupling of soliton motion and plasma oscillations in these systems. In experiments with mesa structures on single crystals of $Bi_2Sr_2CaCu_2O_{8+x}$, they have shown that the $I-V$ characteristics and microwave emission data can be explained by collective motion of Josephson vortices in the intrinsic junctions. There is a broadband microwave emission signal that does not obey the Josephson relation in the high-magnetic-field regime. They explain this by plasma oscillations excited through Cherenkov coupling that are excited by Josephson vortices.

The propagation of solitons in JTLs appears in several fields of nonlinear physics, including superconductivity. The recent increased interest in this field is fueled by new potential applications, including high-temperature superconductors. A soliton in a JTL is often called a "fluxon" since it accounts for a magnetic flux quantum moving between two superconducting electrodes. The single fluxon dynamics in discrete JTLs differs essentially from that in continuous lines as a result of strong interaction with small-amplitude linear waves. Ustinov reviews recent progress in applications, experiments, and modeling of soliton (fluxon) dynamics in mutually coupled continuous lines (stacked, long Josephson junctions) and in discrete JTLs [78]. The magnetic flux quanta in Josephson junctions, often called fluxons, in many cases behave as solitons.

The ω-phase in certain elements (for example, zirconium) and alloys (for example zirconium–niobium) is a metastable state and usually coexists with the beta body-centered cubic (bcc)-matrix in the form of small particles. The bcc δ to ω-phase transformation is induced either by quenching or applying pressure. Sanati and Saxena [74] studied the formation of domain walls in these materials by extending the Cook–Landau model for the ω-phase transformation by including a spatial gradient (Ginzburg) term of the scalar order parameter. They obtained a static equilibrium condition for an asymmetric double-well Landau free-energy potential and obtained different quasi-one-dimensional kink-type soliton-like solutions corresponding to three different types of domain walls between the ω-phase and the beta matrix. They also calculated the formation energy and the asymptotic interaction between domain wall soliton lattice solutions.

Habib et al. [27] study the thermodynamics of 1+1-dimensional classical thermodynamics where at certain temperatures, the Schrödinger-like equation resulting from the transfer integral method to compute the partition function is quasiexactly solvable. Consequently, at these temperatures the partition and correlation functions can be calculated exactly, both above and below the short-range order transition. They apply the analysis to the hyperbolic analog of the well-known double sine-Gordon system and to the double sinh-Gordon model and make an important observation connecting the stationary solutions with the corresponding solutions in the Landau–Ginzburg and the double sine-Gordon theories. Furthermore, the resulting probability distribution functions and correlation lengths from high resolution Langevin simulations are in striking agreement with the exact solutions of the transfer integral.

Systems of macroscopic particles, such as sand or powders, exhibit complex behavior despite their apparent simplicity. Vibrated sand can segregate according to the size of the grains and display rich patterns, solitary waves, or convection rolls. Even though the individual grains are solid, collectively they exhibit both solid-like and liquid-like properties. As the granular system relaxes, a growing number of beads have to be rearranged to enable a local density increase. The time scale associated with such events increases exponentially, and the final state is approached logarithmically slowly as $1/\log t$. The density of a vibrated granular material relaxes from a low-density initial state into a higher-density final steady state according to an inverse logarithmic law. Ben-Naim et al. [7] use a simple theoretical model of granular compaction to capture the essential mechanism underlying this remarkably slow relaxation. The steady-state fluctuations in the model are similar to the experimentally observed ones and exhibit a logarithmically slow approach to the final state.

There is a dynamical transition in correlated, driven diffusion of a two-dimensional array of particles driven by a constant force in the presence of a periodic external potential. The system exhibits a hierarchy of dynamical phase transitions when the driving force is varied. In his poster at NWSPS, Dauxois used a simple phenomenological approach to reduce the system of strongly interacting particles to weakly interacting quasiparticles (kink). This strongly coupled system displays a hysteretic behavior even at nonzero temperature and can be viewed as a first step toward understanding nanotribology.

In the nearly linear regime of a material with a weakly disordered potential, the mass of solitons whose incident mass is relatively small compared with their velocity decays exponentially to approach a constant after a large number of scattering events. However, when the ratio of the mass of an incident soliton to the incident soliton velocity is sufficiently large, the mass of the solitary wave asymptotically approaches a constant, while the velocity decays slowly to zero. This behavior agrees with theoretical predictions and the numerical study presented by Bronski [11]. In simulations, he followed the soliton through a large number of scattering events by calculating the dynamics in the locally stationary frame of the soliton.

5. Pattern formation

Many natural phenomena result in the formation of patterns. Chemical reactions, evolution of biological systems, free surface flows, fluid convection, to name just a few, often produce beautiful geometric figures, in general associated with an underlying symmetry. It is one of the intriguing facts of nature that mathematical models capable of describing the formation and evolution of patterns share a "universal" structure, one that it is somewhat independent of the specific physical process that generates the pattern. Amplitude equations like the celebrated complex Ginzburg–Landau (CGL) system keep appearing everywhere in the study of pattern formation, due to the fact that often the pattern is the long-wave modulation of problem-specific dynamics that occurs at shorter length scales. This dynamics can be incredibly complex, yet the large scale evolution through an amplitude model can be associated with just a few order parameters, which pin down, besides a particular figure, when bifurcations among patterns are going to occur.

The paper by Bowman et al. [9] addresses one of the outstanding issues in the area of amplitude equations for convection patterns, that of the existence of "defects" (e.g., dislocations in the striped structure of convection rolls). They derive evolution equations for the pattern wavenumber through its phase function and show that they have the mixed hyperbolic–parabolic PDE structure of nonlinear advection plus diffusion equations. By examining the stationary weak solutions of the hyperbolic part of the equation for the phase function and studying the effects of the diffusive operator as a pertubation, an interesting picture of the occurrence of defects emerges. Defects are associated with the weak-solution shocks of the hyperbolic phase equation, and the criterion for selecting which weak solutions are realized in a given configuration is offered by the regularization of the diffusive term. Bowman et al. view defects as jumps in an otherwise slowly varying wavevector function. This is appropriate whenever the global pattern closely approximates a periodic wave. A more microscopic viewpoint is, however, necessary whenever a pattern is not so ordered. Defects can sometimes be identified as regions of high curvature along fronts, sharp transitions between different states of a certain material or different reacting substances. Moving fronts are among the most common elements of patterns observed in nature. Front instabilities are often responsible for the formation of a certain global pattern over a long time scale, or their indefinitely repeating occurrence can lead to a spatial and temporal chaotic state of a system. The traditional approach to a theoretical description of front instabilities is to derive evolution equations that relate the local velocity of a front to its geometric curvature.

The paper by Hagberg and Meron [28] in this special issue addresses a situation, near the onset of a certain bifurcation, where the front velocity can no longer be modeled by a simple function of the front curvature. They derive a system of integro-differential equations that relates the velocity to the curvature through a *nonlocal* equation for the bifurcation order parameter, within the approximation of slow propagation and weak curvature. Hagberg and Meron test their equations vs. numerical simulation of the FitzHugh–Nagumo model, obtaining excellent quantitative agreement for the regimes of the model's derivation, when spontaneous nucleation of spiral waves and transitions between counterpropagating fronts occur.

When the dominant dispersion in the KdV-like model

$$u_t + (u^2)_x + \eta u_{xxx} = u_{xxxxx} \tag{4}$$

is the quintic term, Hyman and Rosenau observed that pulsating multihumped solitary waves, called multiplets, are spontaneously created from a wide initial pulse [36]. These quasistable pulsating waves collide nearly elastically with other multiplets and propagate in a breather-like fashion with a variable modulated speed. As a multiplet propagates, it radiates a small amount of energy behind an oscillating tail. Eventually, the energy loss drains the bound holding the solitary waves together and a single solitary wave breaks free from the multiplet, reducing the number of pulsating peaks by one.

In recent years Rosenau has also studied a wide range of KdV-like systems with a variety of nonlinear dispersions and diffusions mechanisms. In this special issue [72] he continues his investigation by deriving a remarkable variety of exact solutions for

$$u_t + a(u^m)_x + (u^n)_{xxx} = \mu(u^k)_{xx}. \tag{5}$$

The most notable of the cases is $m = k + 1 = n + 2$ when there is a detailed balance among the nonlinear mechanisms. Here the spatial patterns are independent of the amplitude, and there are explicit solutions for the traveling waves. Furthermore, the special case $a = (2\mu/3)^2$ can be mapped into a linear equation exhibiting rational, periodic or aperiodic solutions.

The subject of pattern formation is firmly rooted in experimental observations. The paper by Lauterbach et al. [51] presents an experimental study of great importance for the manufacturing of efficient catalytic converters, where an oxidation reaction takes place along surfaces coated with Palladium. By using microlytography the investigators are able to control the creation of differently shaped domains of thin Palladium layers on a Titanium substrate. Their paper provides several observations on how the formation of reaction–diffusion patterns, like spiral waves, moving fronts etc., are affected by the domains' shapes and symmetries. These observations provide strong motivation for the development of a model capable of predicting the outcome of an experiment with a given domain, a necessary step towards mastering the effects of domain geometries for a more efficient catalytic reaction.

6. Summary

In this work we gave a brief overview of both the papers in this special issue and some of the presentations at the NWSPS Conference. Unfortunately, we could only touch on the complex and compelling nature of nonlinear waves within the classical physical systems of fluids, optics materials. The recent advances in these physical sciences are also helping to build synergetic paradigms in nonlinear science that one day will cover these problems as special cases and may be extended to new situations in biology, sociology, economics, epidemiology, ecology, physiology and the other sciences. The nonlinearities in these fields create patterns of social collective behavior which can be analyzed from the nonlinear wave viewpoint. Already in biology, the applications of nonlinear waves in pattern formation and selection is being applied on scales from solitons in alpha-helix proteins to reaction–diffusion systems modeling the colorful patterns of strips and spots on zebras and giraffes.

The theoretical understanding of nonlinear waves and solitons has built a framework for understanding nonlinear systems at a fundamental physical and mathematical level. This level of understanding is now deeply affecting technical applications. Nonlinear waves analysis has become a paradigm for understanding nonlinear materials, fluid flows, nonequilibrium systems with diffusion, etc.

Of course, there are still many fundamental questions and challenging problems lying ahead. These challanges include: a better understanding of the creation of nonlinear wave patterns and the dynamics and competition among the patterns; the role of symmetry in pattern selection and investigating the dynamics of the solution seeking local minima in energy space and determining the basins of attraction; rigorously assessing the validity of the various solutions in an approximating hierarchy of equations, such as from Navier–Stokes to Green–Naghdi to KdV to ODEs. Also, can this viewpoint help to understand the formation of patterns and intermittency in turbulent flows? There are applications in nonlinear optical transmission lines, predicting global climate change, understanding complex chemical patterns that will capture the imagination and efforts nonlinear scientists for years to come.

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